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Comparative Study of Arakawa Scheme and WENO Scheme for Solving Advection Equation

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Abstract

In this paper, we investigate the impact of Arakawa and WENO advection schemes in numerical ocean modelling. Numerical simulation s in ocean models rely on non-oscillatory advection scheme, which minimizes numerical dissipation and dispersion problems. We study the Arakawa scheme and WENO-5 scheme in the 2-D advection problem. The Arakawa scheme preserves important physical phenomena and the WENO-5 scheme emphasizes high resolution and non-oscillatory solutions, so we study both numerical methods. Our key focus is to observe the oscillatory behavior of the numerical solutions, examine the total variation diminishing (TVD) property and assess numerical dissipation and dispersion of these schemes . We observe that the Arakawa scheme shows an oscillatory solution with slight dissipation, yet preserves energy well. The WENO-5 scheme produces a non-oscillatory numerical solution and minimizes numerical dissipation in 2-D advection equations. The WENO-5 scheme performs better than Arakawa 2-D advection scheme and the WENO-5 reconstruction minimizes numerical dissipation.

Keywords: Ocean modelling, Arakawa scheme, WENO scheme, total variation, TVD.

1. Introduction

Ocean dynamics has a significant impact on the Earth's climate system. The numerical ocean modeling is a core tool for understanding the ocean dynamics, the patterns of water movement in the ocean. The study of ocean modeling also includes how the temperature, salt and pollution are transported through the ocean water. Scientists use numerical ocean modeling to predict ocean behavior, such as a rise in sea level and changes in ocean eddies that involve cyclones and anticyclones. We aim to investigate ocean models that produce accurate numerical simulations of complex and turbulent flows. This demands advection schemes with high performance and the smallest dissipation. The advection equation plays a vital role in such research, as the advection process describes the transportation of heat, salt and other components by water flow in ocean. The simplest form is the one-dimensional advection equation (section 10.1 in [5]):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

The traditional advection scheme often exhibits significant numerical dissipation and

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dispersion, sometimes with oscillatory behavior in solutions. It is very challenging to develop an accurate numerical advection scheme that minimizes both dissipation and dispersion. In our research, we investigate numerical methods - the Arakawa scheme and the WENO-5 scheme to solve 2-dimensional advection equations. We focus on observing the oscillatory behavior in the numerical solutions, accuracy, stability and total variation.

Researchers have been investigating many numerical schemes to solve the advection equation. Some established methods are the Upwind scheme (first order scheme), Lax- Wendroff scheme (second order scheme) [5]. The upwind scheme performs with low accuracy and huge dissipation and the Lax-Wendroff scheme performs well with low dissipation but the solution is oscillatory [5].

Nowadays, mathematicians are interested in investigating non-oscillatory and low dissipative schemes to solve the advection equation, explaining ocean dynamics. Weighted Non-Oscillatory (WENO) schemes are non-oscillatory schemes that avoid excessive numerical dissipation [6]. In this report, our primary focus is to study the WENO schemes in detail and examine the properties and behavior of the numerical solutions, particularly their non-oscillatory and low-dissipation characteristics and compare with ARAKAWA scheme. There exist many works using WENO schemes for solving multidimensional transport equations and simulations for ocean models. A numerical technique is introduced in paper [8] for solving two-dimensional transport problems on a cubed sphere using finite-volume discretization with a WENO-limited scheme. In that paper, they implement a new function-based WENO construction in space differential transform. Another WENO reconstruction is implemented in paper [9] for the rotating shallow water equations. Here, they reconstruct the WENO scheme on the mass flux and on the nonlinear Coriolis term. This discretization preserves the potential vorticity and produces numerical solution with low energy dissipation, well-implementation of WENO scheme. A new WENO reconstruction is presented in [12] for simulation of a momentum advection scheme as ocean model that avoids excessive numerical dissipation. In their work, they rewrite the momentum advection discretization in the rotational form of WENO advection reconstruction, transforming the vertical advection into a combination of divergence flux and conservative vertical flux to employ upwind reconstruction and avoid dispersive grid-scale noise. These approaches are entirely new, as old works employ center differencing in momentum advection discretization and fail to upwind construction in vertical advection [12].

In this paper, we study total variation and TVD, a method for the space discretization of Jacobian in the 2-D advection equation and a deep study on the WENO schemes.

In the next section we are going to discuss the governing equation,

1.1 Governing Equation

The 2-D advection equation is

$$\frac{\partial \gamma}{\partial t} + u\frac{\partial \gamma}{\partial x} + v\frac{\partial \gamma}{\partial y} = 0 \tag{2}$$

along with

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

This 2-D advection equation preserves the integral conservation property. i.e. (chapter 1 of [4])

$$\frac{d}{dt} \int_{\Omega} \gamma^{p} dx \ dy = 0$$

2. Numerical Schemes for the Advection Equation

In this section, we discuss a finite difference (FD) Arakawa scheme and a finite volume WENO-5 scheme for the advection equation. FD schemes are simple in implementation, fast and adapt to complex boundary conditions [7]. In the finite difference method, the derivatives are approximated by finite differences at each point, which can lead to errors in the numerical solution near discontinuities (section 1.2 in [4]). But the finite volume method effectively works in the computation of discontinuous problems (section 1.2 in [4]). We also discuss the importance of the total variation and TVD in numerical solutions.

2.1 Total Variation and TVD

The Lax-Wendroff scheme (a second-order scheme) produces oscillations near discontinuity [5], but we want to investigate schemes with second-order accuracy that ensure no nonphysical oscillations will arise. The total variation (TV) of a function is a way to measure the oscillations in the numerical solution. For an arbitrary function $q(x)$ the TV is defined as (section 6 .7 in [4]):

$$TV(q) = \sum |q(\xi_j) - q(\xi_{j-1})| \tag{4}$$

where the supremum is taken over the sub-intervals on the real line $-\infty = \xi 0 < \xi 1 < \xi 2 < ... < \xi N = \infty$.

For the grid function Q (section 6.7 in [4]):

$$TV(Q) = \sum |Q_i - Q_{i-1}| \tag{5}$$

If the numerical solution of the advection diffusion equation is non-oscillatory, then the total variation of Q^n, decreases with time. To ensure that the method does not increase the total variation to avoid oscillations, the method must satisfy the total variation diminishing (TVD) property.

“A two-level method is called total variation diminishing (TVD) if, for any set of data Q^n, the values Q^{n+1} computed by the method satisfy” (section 6.7 in [4]):

$$TV(Q^{n+1}) \leq TV(Q^n) \tag{6}$$

Here, diminish does not mean strictly decreasing; it also may be constant with respect to time. An appropriate term may call as total variation non increasing (TVNI) was used in the original work of Harten [2]. The simple flux limiter method, such as the minmod limiter is TVD (section 6 .9 in [4]). For a more complicated limiter scheme, the following Harten theorem [2] can be used to mathematically prove that the limiter function is TVD. The theorem is proved for periodic boundary conditions.

Proposition 1 *(Harten Theorem) Consider a general method of the form*

$$Q_i^{n+1} = Q_i^n - C_{i-1}\left(Q_i^n - Q_{i-1}^n\right) + D_i\left(Q_{i+1}^n - Q_i^n\right) \quad (7)$$

over one time step, where the coefficients C_n^{i+1} and D_{n-1}^i are arbitrary values (which in particular may depend on the values of Q^n in some way, that is, the method may be non-linear). Then

$$TV\left(Q^{n+1}\right) \le TV\left(Q^n\right)$$

provided the following conditions are satisfied:

$$C_{i-1}^n \ge 0 \;\; \forall \;\; i \quad (8)$$

$$D_i^n \ge 0 \;\; \forall \;\; i \quad (9)$$

$$C_i^n + D_i^n \le 1 \;\; \forall \;\; i \quad (10)$$

(Section 6.12 *in [4])*

Proof: We know that the total variation is defined as:

$$TV = \Sigma \left|Q_{i+1}^n - Q_i^n\right| \quad (11)$$

Now consider:

$$Q_{i+1}^{n+1} - Q_i^{n+1} = \left[Q_{i+1}^n - C_i^n\left(Q_{i+1}^n - Q_i^n\right) + D_{i+1}^n\left(Q_{i+2}^n - Q_{i+1}^n\right)\right] - \left[Q_i^n - C_{i-1}^n\left(Q_i^n - Q_{i-1}^n\right) + D_i^n\left(Q_{i+1}^n - Q_i^n\right)\right]$$
$$= \left(Q_{i+1}^n - Q_i^n\right)\left(1 - C_i^n - D_i^n\right) + D_{i+1}^n\left(Q_{i+2}^n - Q_{i+1}^n\right) + C_{i-1}^n\left(Q_i^n - Q_{i-1}^n\right)$$

Applying the triangle inequality in above equation, we get:

$$\left|Q_{i+1}^{n+1} - Q_i^{n+1}\right| \le \left|Q_{i+1}^n - Q_i^n\right|\left|1 - C_i^n - D_i^n\right| + D_{i+1}^n\left|Q_{i+2}^n - Q_{i+1}^n\right| + C_{i-1}^n\left|Q_i^n - Q_{i-1}^n\right|$$

Summing all i and the inequality above can be written as:

$$\sum_i \left|Q_{i+1}^{n+1} - Q_i^{n+1}\right| \le \sum_i \left|Q_{i+1}^n - Q_i^n\right|\left|1 - C_i^n - D_i^n\right|$$
$$+ \sum_i D_{i+1}^n\left|Q_{i+2}^n - Q_{i+1}^n\right| + \sum_i C_{i-1}^n\left|Q_i^n - Q_{i-1}^n\right|$$

Now change the index in the last two sums for periodic boundary conditions:

$$\sum_i D_{i+1}^n\left|Q_{i+2}^n - Q_{i+1}^n\right| = \sum_i D_i^n\left|Q_{i+1}^n - Q_i^n\right|$$
$$\sum_i C_{i-1}^n\left|Q_i^n - Q_{i-1}^n\right| = \sum_i C_i^n\left|Q_{i+1}^n - Q_i^n\right|$$

Since the condition (8) is satisfied. We can write:

$$\sum_i \left|Q_{i+1}^{n+1} - Q_i^{n+1}\right| \leq \sum_i \left(\left(1 - C_i^n - D_i^n\right) + C_i^n + D_i^n\right)\left|Q_{i+1}^n - Q_i^n\right| \leq \sum_i \left|Q_{i+1}^n - Q_i^n\right| \qquad (12)$$

Using the definition (11), we can write (12) as:

$$TV\left(Q^{n+1}\right) \leq TV\left(Q^n\right)$$

This completes the proof.

For example, the upwind scheme, we can rewrite in the form of TVD [5]:

$$Q_i^{n+1} = Q_i^n - C_{i-1}^n\left(Q_i^n - Q_{i-1}^n\right)$$

for the choice $C_{i-1}^n = \frac{a\Delta t}{\Delta x}$ and $D_i^n = 0$. We can easily observe using CFL condition: $0 \leq C \leq 1$, that $C_{i-1}^n \geq 0$, $D_i^n \geq 0$ and $C_i^n + D_i^n \leq 1$. So, the conditions from (8) to (10) are satisfied. So, the upwind scheme is TVD. Similarly, we rewrite the Lax-Wendroff method as [5]:

$$u_m^{n+1} = \left[1 - \left(\frac{a\Delta t}{\Delta x}\right)^2\right]u_m^n - \frac{1}{2}\left[\frac{a\Delta t}{\Delta x} + \left(\frac{a\Delta t}{\Delta x}\right)^2\right]u_{m-1}^n + \frac{1}{2}\left[\left(\frac{a\Delta t}{\Delta x}\right)^2 - \frac{a\Delta t}{\Delta x}\right]u_{m+1}^n. \qquad (13)$$

Equation (13) is equivalent to the form of equation (7):

$$Q_i^{n+1} = Q_i^n\left(1 - C_{i-1}^n - D_i^n\right) - C_{i-1}^n Q_{i-1}^n + D_i^n Q_{i+1}^n$$

for the choice $C_{i-1}^n = \frac{1}{2}\left[\frac{a\Delta t}{\Delta x} + \left(\frac{a\Delta t}{\Delta x}\right)^2\right]$ and $D_i^n = \frac{1}{2}\left[\left(\frac{a\Delta t}{\Delta x}\right)^2 - \frac{a\Delta t}{\Delta x}\right]$.

But from CFL condition $\left|\frac{a\Delta t}{\Delta x}\right| \leq 1$, we find $D_i^n \leq 0$ which violates the condition (9) of TVD. So, the Lax-Wendroff is not a TVD method. We will observe the TVD property of ARAKAWA and WENO -5 schemes later in this paper.

2.2. Arakawa Scheme

We consider the scalar field $\gamma(x, y, t)$, where the advection is carried out by an incompressible 2-D velocity field (u, v), which is derived from the stream function $\psi(x, y)$ as follows:

$$u = -\frac{\partial\psi}{\partial y} \qquad (14)$$

$$v = \frac{\partial\psi}{\partial x} \qquad (15)$$

Using the definition of the Jacobian, we obtain:

$$J(\psi, \gamma) = \left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\gamma}{\partial y}\right) - \left(\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\gamma}{\partial x}\right) \qquad (16)$$

Then, the two-dimensional advection equation can be written as:

$$\partial\gamma/\partial t + J(\psi, \gamma) = 0 \qquad (17)$$

We study the numerical scheme for this 2-D advection equation following the work in [1]. According to [1], the space discretization of Jacobian $J(\gamma, \psi)$ are given by

$$J(\gamma, \psi) = \frac{1}{3}(J_1 + J_2 + J_3)$$

where,

$$J_1 = \left(\frac{1}{4\,\Delta x \Delta y}\right)\left[(\gamma_{i+1,j} - \gamma_{i-1,j})(\psi_{i,j+1} - \psi_{i,j-1}) - (\gamma_{i,j+1} - \gamma_{i,j-1})(\psi_{i+1,j} - \psi_{i-1,j}) \right]$$

$$J_2 = \left(\frac{1}{4\,\Delta x \Delta y}\right)\left[\gamma_{i+1,j}(\psi_{i+1,j+1} - \psi_{i+1,j-1}) - \gamma_{i-1,j}(\psi_{i-1,j+1} - \psi_{i-1,j-1}) - \gamma_{i,j+1}(\psi_{i+1,j+1} - \psi_{i-1,j+1}) + \gamma_{i,j-1}(\psi_{i+1,j-1} - \psi_{i-1,j-1}) \right]$$

$$J_3 = \left(\frac{1}{4\,\Delta x \Delta y}\right)\left[\gamma_{i+1,j+1}(\psi_{i,j+1} - \psi_{i+1,j}) - \gamma_{i-1,j-1}(\psi_{i-1,j} - \psi_{i,j-1}) - \gamma_{i-1,j+1}(\psi_{i,j+1} - \psi_{i-1,j}) + \gamma_{i+1,j-1}(\psi_{i+1,j} - \psi_{i,j-1}) \right]$$

For time discretization, we follow the R-K second-order approximation. Predictor step (compute intermediate state):

$$\tilde{\gamma}_{i,j} = \gamma_{i,j}^n + \Delta t \cdot J_{i,j}^n \tag{18}$$

Evaluation of the Jacobian at the predicted step:

$$\tilde{J}_{i,j} = J(\tilde{\gamma}, \psi)_{i,j} \tag{19}$$

The fully discretized scheme is:

$$\gamma_{i,j}^{n+1} = \gamma_{i,j}^n + \left(\frac{\Delta t}{2}\right)(J_{i,j}^n + \tilde{J}_{i,j})$$

This scheme preserves the property of anti-symmetry, energy conservation and momentum conservation with time [1].

2.3. The WENO Schemes

An efficient approach to solving the hyperbolic and convection–diffusion equations with discontinuous solutions is weighted essentially non-oscillatory (WENO) schemes [11]. These schemes also work on solutions with sharp gradient regions. WENO was introduced by Harten in [3], aiming to achieve high-order accuracy in smooth regions and solve shocks or essentially non-oscillatory discontinuity modes. In this section, we study the WENO construction for the advection scheme and discuss their numerical property.

2.4 WENO-5 Scheme

The main idea is to use all the candidate stencils for polynomial interpolation or reconstruction rather than using only one in WENO approach [11]. The updated WENO schemes were introduced in [6], which presented a WENO reconstruction in the finite volume approach to solve hyperbolic conservation laws. A more general approach is to produce a $(2k + 1)$ th-order WENO scheme to solve the multi-dimensional conservation laws.

The basic idea of WENO is to form a convex combination of the three candidate polynomials $p_1(x),\ p_2(x),\ p_3(x)$ where the polynomials are constructed on the following three stencils respectively:

$$S^1=\{I_{i-2},I_{i-1},I_i\}$$
$$S^2=\{I_{i-1},I_i,I_{i+1}\}$$
$$S^3=\{I_i,I_{i+1},I_{i+2}\}$$

The reconstruction at the right boundary of cell I_i is determined by:

$$u^-_{(i+1/2)} = w^1 p^{(1)}\left(x_{i+1/2}\right) + w^2 p^{(2)}\left(x_{i+\frac{1}{2}}\right) + w^3 p^{(3)}\left(x_{i+\frac{1}{2}}\right)$$

This crucial ingredient is the choice of the combination coefficients, called the non-linear weights and they satisfy

$$w_i \geq 0, \sum_{l=1}^{3} w_l = 1$$

These non-linear weights are designed to reflect the smoothness of the function $u(x)$ in each stencil $S^{(l)}$. The non-linear weights depend on linear weights γ_l and smoothness indicators β_l, which are the scaled square sum of the $L2$- norms of derivatives of the reconstruction polynomial $p^{(l)}(x)$ over target cells. According to [11] the values of the linear weights and the explicit form of the smoothness indicators β_l are given by:

$$\gamma_0=\frac{1}{10},\gamma_1=\frac{6}{10},\gamma_2=\frac{3}{10}$$

and

$$\beta_0= \frac{13}{12}(u_{i-2} - 2u_{i-1}+ u_i)^2+\frac{1}{4}(u_{i-2} - 4u_{i-1} + 3u_i)^2$$

$$\beta_1= \frac{13}{12}(u_{i-1} - 2u_i+ u_{i+1})^2+\frac{1}{4}(u_{i-1} -u_{i+1})^2$$

$$\beta_2= \frac{13}{12}(u_i - 2u_{i+1}+ u_{i+2})^2+\frac{1}{4}(3u_i - 4u_{i+1} + u_{i+2})^2$$

With linear weights and the smoothness indicators the non-linear weights are defined as:

$$\omega_k = \frac{\widetilde{w}_k}{\widetilde{w}_j} \qquad \text{where} \qquad \widetilde{w}_k = \frac{\gamma_k}{\left(\varepsilon + \beta_k\right)^2}$$

and $\varepsilon\left(=10^{-6}\right)$ is a small positive number to avoid the zero denominator. The specific choice of the weights is made to maintain two vital properties. One of the properties is, if the solution is smooth enough in all candidate stencils, the non-linear weights should be very close to the linear weights γ_i. This choice of ω_i preserves the highest possible order accuracy from the combined stencils. The other property is that if only one stencil contains a discontinuity, then a small weight should be assigned to that stencil with the discontinuity. This choice ensures the effect of the weight in the approximation is negligible. The advection equation is a time-dependent problem and we need time accuracy as well. In paper [10], a class of multilevel-type high-order time discretization is presented that holds TVD. Following the method, we discretize the time derivative called the strong stability preserving (SSP) time discretization as follows:

$$u^1 = u^n + \Delta t \cdot L(u^n)$$

$$u^2 = \frac{3}{4}u^n + \frac{1}{4}\left(u^1 + \Delta t \cdot L\left(u^1\right)\right)$$

$$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}\left(u^2 + \Delta t \cdot L\left(u^2\right)\right)$$

where,
$$L(u) = -\left(\frac{1}{\Delta x}\right)\left(\hat{f}_{\left(i+\frac{1}{2}\right)} - \hat{f}_{\left(i-\frac{1}{2}\right)}\right)$$

3. Numerical Examples

In this section, we present numerical solutions of the 2-D advection equations based on the finite difference and finite volume schemes. We also studied in Section 2 under various physical parameters, including advection speed, grid resolution and final time. Python is used as the computational platform for all numerical experiments. To test the 2-D advection equation, we consider the periodic domain $\Omega=[0,\ 1]\times[0,\ 1]$ with grid sizes $N_x \times N_y = 200\times 200$ and temporal grid size $N_t = 600$. We consider the Gaussian function as the initial condition of the form:

$$\gamma(x,\ y,\ 0) = \exp\left(-100\ (x - 0.5)2 + (y - 0.5)2\right)$$

and the stream function of the form:

$$\psi(x,y)=\sin\left(\frac{2\pi x}{L_x}\right)\sin\left(\frac{2\pi y}{L_y}\right)$$

The numerical solution is obtained for $t = 3$ with $\Delta t = 0.005$ and the spatial intervals are $L_x=1\ .0$ and $L_y=1.0$ with $\Delta x=\frac{1}{N_x}$ and $\Delta y=\frac{1}{N_y}$.

We study two numerical schemes for the two-dimensional cases: a second-order finite difference scheme (Arakawa scheme) [1] and the fifth-order WENO-5 scheme in the finite-volume approach [11]. We study a finite volume method as it performs better than finite difference method in some hyperbolic problems to produce conservative and high resulation, non-oscillatory solutions.

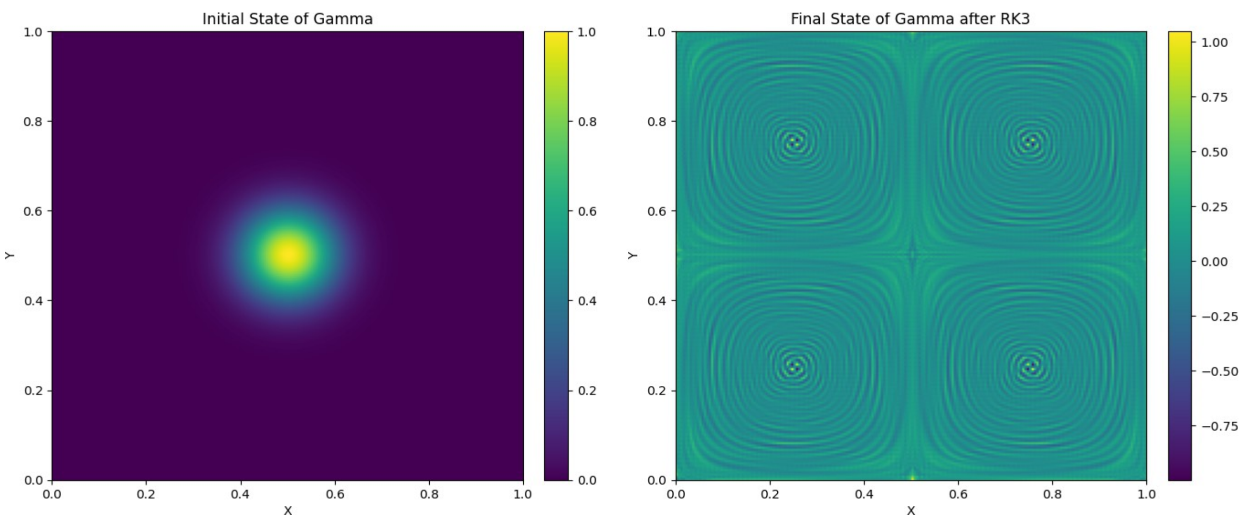

Figure 1: Advection of a 2-D Gaussian scalar field $\gamma(x,\ y,\ t)$ with R-K3 time discretization. The vortex flow obtained by the Arakawa scheme at $t = 3$ after initialization with $\Delta t = 0.005$.

Figure 1 shows the vortex flow of a two-dimensional Gaussian scalar field at $t = 3$ after initialization obtained by the Arakawa scheme, which is a second-order scheme. The left plot shows the smooth Gaussian blob centered in the domain at $(x,\ y) = (0.5,\ 0.5)$. The plot on the right panel shows the final state of the scalar field $\gamma(x,\ y,\ t)$ and we observe that the initial Gaussian blob is stretched and divided into a set of vortex flows. It is obvious as we consider the stream function $\psi(x,\ y)$ to compute the advection speeds u-in the direction of x and v- in the direction of y. We also observe that the scalar is well transported by the advection process along the streamline. The spirals indicate periodicity and symmetry of the vortex flows. But the numerical solution shows oscillations, as the scheme is second-order. This indicates numerical dispersion as those schemes may be calculating the advection speed slightly wrong. The 2-D advection equation has the integral conservation property and we find $\frac{d}{dt} \int_{\Omega} \gamma^{p} dx \ dy = 0$ using the Arakawa scheme for $p = 1$, which evidences the scheme preserves the integral. Another important factor is the Jacobian discretization for space derivatives $J(\psi, \gamma)$ in (14) should preserve the anti-symmetry property i.e. $J(\gamma, \psi) = -J(\psi, \gamma)$. We find that our implementation according to the scheme preserves anti-symmetry within tolerance $(10-12)$ and our scheme calculates the difference $J(\gamma, \psi) + J(\psi, \gamma)$ is $(10-16)$ less than the tolerance level, which is evidently enough to state the property preservation.

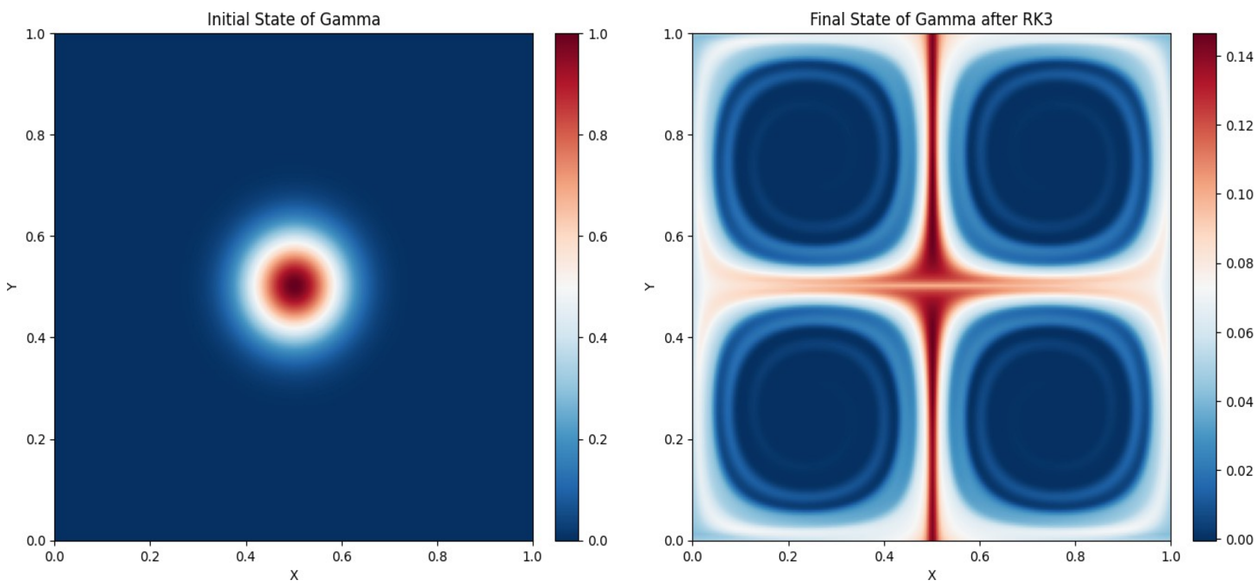

Figure 2: Advection of a 2-D Gaussian scalar field $\gamma(x, y, t)$ with SSP time discretization. The vortex field is obtained by WENO-5 scheme (Finite Volume) at $t = 1$ after initialization with $\Delta t = 0.005$ and $N_t = 200$.

Figure 2 shows the advection of a 2-D Gaussian scaler $\gamma(x, y, t)$ at $t = 3$ obtained by the WENO-5 scheme and SSP time discretization. We observe the initial Gaussian blob spreads and reforms into a set of spiral streamlines. We notice that the vortex flows along the streamlines are non-oscillatory, which is the main goal of the WENO-5 scheme, to produce non-oscillatory solutions. But still, we notice the sudden fall of the scalar indicates the loss of amplitude for the scheme despite being a higher-order scheme. We also present the visualization of the solution of the WENO-5 scheme for long time simulation.

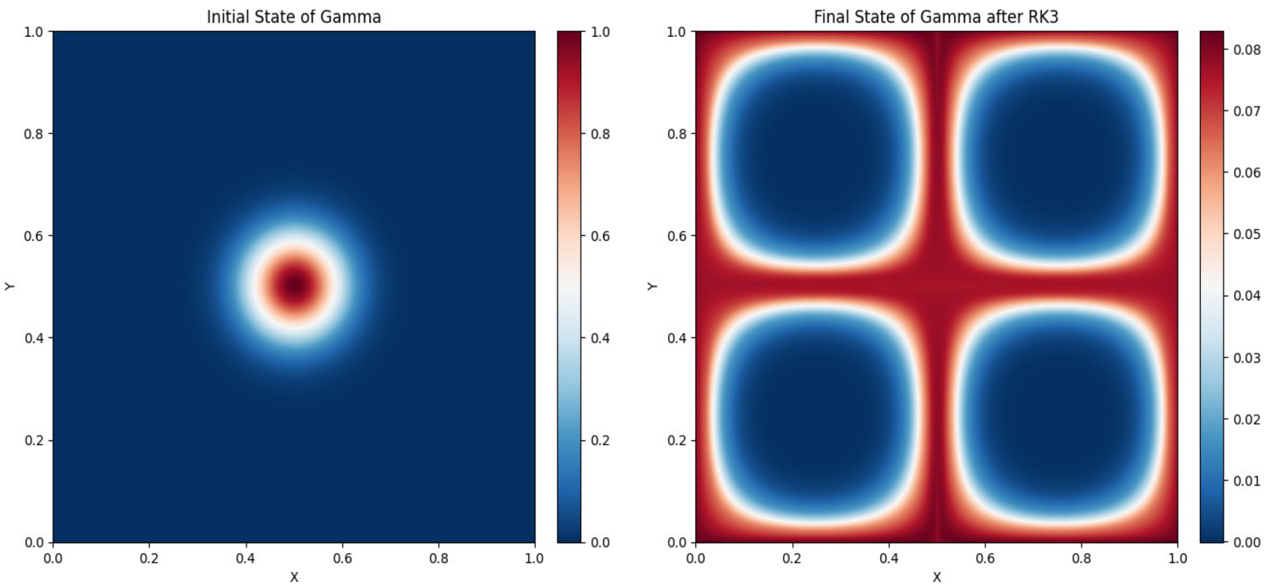

Figure 3: Advection of a 2-D Gaussian scalar field $\gamma(x, y, t)$ with SSP time discretization. The vortex field is obtained by WENO-5 scheme (Finite Volume) at $t = 3$ after initialization with $\Delta t = 0.005$.

Figure 3 represents the numerical results for the WENO-5 scheme with 2-D Gaussian scaler field at $t = 3$. We observe that in the long time the scheme also produces non-oscillatory flow and spreads the initial blob accurately into the spirals, which indicates the vortex flows along the streamlines. We also notice numerical dispersion in long-time simulations.

3.1 L-2 Norm Error

We compare the L2-norm of solution to the 2-D advection equation obtained using the Arakawa and the WENO-5 scheme. Since the exact solution of 2-D advection equation is unavailable, the L-2 norm is used to evaluate the numerical behavior of the solution. For 2-D cases, we calculate the L2-norm of the solution of the following form:

$$\|u^n(x, t)\|_2 = \sqrt{\sum_{i=1}^{N} |u_i^n|^2 \ \Delta x \ \Delta y}$$

In Figure 4, the L2-norm of solution is observed for the Arakawa scheme. The error grows linearly and slowly over time, reflecting the fact that the scheme preserves stability and the numerical dissipation is comparatively low. In terms of the L2-norm of solution, the scheme almost preserved the norm up to the spatial discretization level, but we see a slight drift due to time discretization error as we applied RK-3 time discretization. Although Figure 5 shows the L2-norm of solution observed for the WENO-5 scheme, which decays over time. Thus, it evidences that the stability of the scheme is well preserved with limited numerical dispersion.

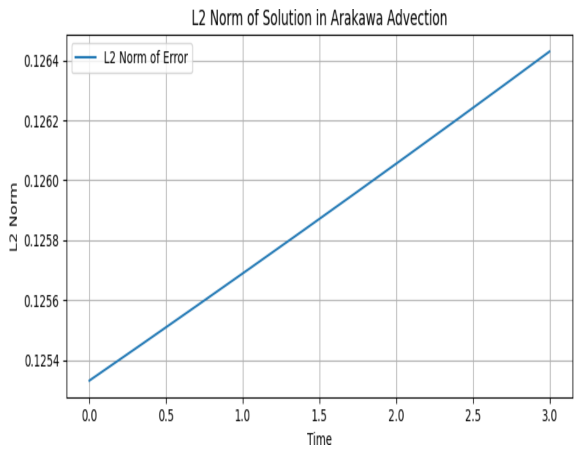

Figure 4: $L2$-Norm of Solution in the Arakawa 2-D advection at $t = 3.0$ with $\Delta t = 0.005$.

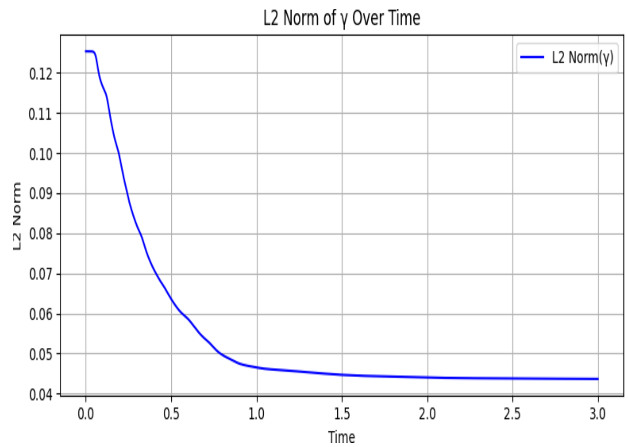

Figure 5: $L2$-Norm of solution in the WENO-5 scheme for 2-D advection at $t = 3.0$ with $\Delta t = 0.005$.

3.3 TVD Property

In this section, we study the TVD property for the 2-D advection equation using the Arakawa and the WENO-5 schemes. We have already obtained that the upwind scheme is TVD, but the Lax-Wendroff fails to maintain TVD in 2.1 and the minmod limiter scheme is a TVD scheme (section 6 .9 in [4]).

In Figure 6, we see the total variance (TV) plot for the numerical solution obtained by the Arakawa scheme and the TV increases slowly with time. This means the scheme is not a TVD scheme, which is expected. We see some oscillations in this plot, which occur due to the spurious oscillation in numerical solution obtained by the Arakawa scheme.

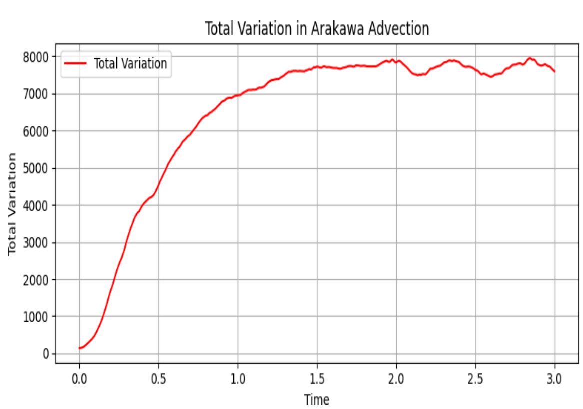

Figure 6: Total Variation for the Arakawa 2-D advection at $t = 3.0$ after initialization.

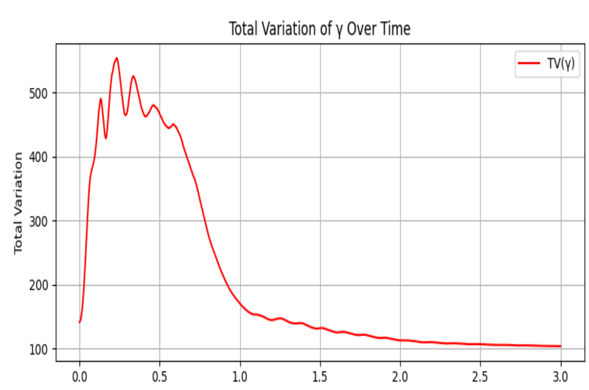

Figure 7: Total Variation of the WENO-5 scheme at $t = 3.0$ after initialization.

Figure 7 illustrates the TV plot for the numerical solution of the 2-D advection equation using the WENO-5 scheme. Initially, we observe that the TV increases with some oscillations, but in the long run it strictly decreases. The WENO-5 reconstruction is not strictly TVD, however, it provides a more stable, high-resolution, and essentially non-oscillatory solution than the Arakawa scheme. Future work will investigate TVD- preserving WENO formulations to further improve the numerical results.

4. Conclusion

We observed the numerical solutions of 2-dimensional advection equations. The comparative analysis of the two-dimensional advection equation included the Arakawa scheme (space discretization of Jacobian) and WENO-5 scheme. Throughout the study, we focused on stability, accuracy, TVD and the physical behavior of the numerical solutions obtained using the Arakawa and WENO-5 methods. We implemented the Arakawa scheme in 2-D advection equation simulation and observed that this scheme showed oscillatory behavior in wave propagation. The WENO-5 scheme performed well in the long time, producing a non-oscillatory solution. Additionally, we analyzed the $L-2$ norm of the solution and total variation for both schemes. The results showed that the WENO-5 scheme better preserved the TVD property, resulting in a more stable and non-oscillatory solution than the Arakawa scheme. In the future, we would investigate further to produce realistic ocean waves with turbulence and extend it to a three-dimensional case using WENO schemes.

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